ESTIMATION OF THE THERMAL RESISTANCE FOR BODIES OF COMPLEX SHAPE

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A rigorous derivation is given of upper and lower bounds for the thermal resistance of a body enclosed between two surfaces, one of which encloses the other. The two-dimensional and three-dimensional cases are examined. Specific formulas are given for the cases of a rectangle in a rectangle and a parallelepiped in a parallelepiped.

1. The problem can be formulated as follows: we wish to find upper and lower bounds for the thermal resistance of bodies of complex shape.

The basic method is based on the analogy between steady thermal and electrostatic fields. Let there be a surface S_0 enclosing a surface S_1. The temperature of S_1 is maintained at $t_1 = 1$, and of S_0 at $t_0 = 0$. The thermal resistance R between surfaces S_0 and S_1 is related to the electrical capacitance C of a capacitor, whose plates, surfaces S_0 and S_1, are maintained at potentials $u_0 = 0$ and $u_1 = 1$, respectively. (The space between S_0 and S_1 is filled with a dielectric with $\varepsilon = 1$.) According to [1],

$$R = \frac{1}{4\pi k} \cdot \frac{1}{C}, \tag{1}$$

where k is the thermal conductivity. We shall evaluate the quantity C^{-1}.

2. What follows is based on the well-known variational determination of capacitance,

$$C = \inf_u \int_D |\nabla u|^2 dV, \quad u|_{S_0} = 0, \quad u|_{S_1} = 1, \tag{2}$$

where inf, the greatest lower bound, is evaluated over the set of differentiable functions transforming into 0 and 1 on S_0 and S_1, respectively. From definition (2) we can derive the following lemma.

Lemma 1. The capacitance is not increased when the system is made symmetrical.

A formal definition of symmetrization and a proof of Lemma 1 are given in [2] and in [3].

Referring the reader to [3] for the proof of Lemma 1, we introduce the definition given there of the concept of symmetrization, used in Lemma 1.

Definition. Symmetrization relative to a plane P transforms a body D into a body D*, which has the properties:

1) D* is symmetrical relative to P;

2) any straight line perpendicular to P and intersecting one of the bodies D or D* also intersects the other; chords intercepted on the straight line by the two bodies have the same length; and

3) the intercept of the straight line considered with D* takes the form of a single section which is bisected by the plane P. An analogous definition holds for symmetrization of the plane of the figure relative to a straight line lying in the plane of the figure. Repeated symmetrization relative to a suitably chosen infinite sequence of planes (lines) transforms any body (figure) into a sphere (circle) with the same volume (area).

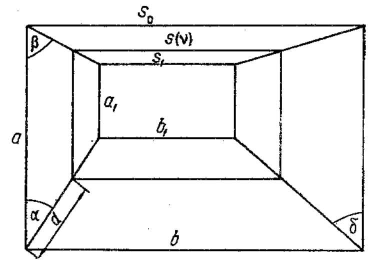

Fig. 1. Rectangle within a rectangle.

The capacitance property indicated in Lemma 1 can be used to find an upper estimate for the quantity C^{-1}. For example, we shall consider the plane problem: to evaluate C^{-1} for a plane situated between curves C_0 and C_1, spanning areas Δ_0 and Δ_1, respectively. It follows from Lemma 1, that

$$C^{-1} \leqslant \ln \frac{\Delta_1}{\Delta_0}, \tag{3}$$

since symmetrization sends the curvilinear ring, bounded by curves C_0 and C_1, into the circular ring for which C^{-1} coincides with the right side of inequality (3). A mathematical proof of Eq. (3) is given in the book [3] and the article [4].

To obtain a lower bound for C^{-1}, we make use of the following inequality:

$$\int_0^{\nu_1} d\nu\, T^{-1}(\nu) \leqslant C^{-1}, \tag{4}$$

where

$$T(\nu) = \frac{1}{4\pi} \iint_{S(\nu)} \frac{d\nu}{dn}\, d\sigma. \tag{5}$$

The function $S(\nu)$ is a family of closed surfaces such that $S(\nu)$ is contained within $S(\mu)$ for $\nu < \mu$, $S(0) = S_0$ and $S(\nu_1) = S_1$. Inequality (4) is given in the book [3], page 85.

3. We shall apply the arguments of §2 to obtain an estimate of the quantity C^{-1}, suitable for practical application. Let the temperature on surface S_1 be 1, and on S_0, 0. From Eq. (3), we have

$$C^{-1} \leqslant \ln \frac{ab}{a_1 b_1}. \tag{6}$$

We shall choose $S(\nu)$ to be the rectangle shown in Fig. 1, and the parameter ν to be the distance noted

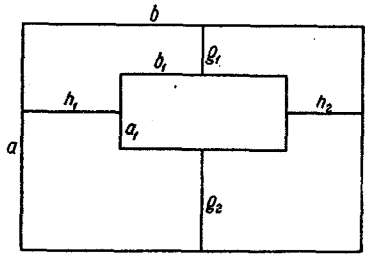

Fig. 2. Rectangle within a rectangle.

on Fig. 1, $0 \le \nu \le d$. We shall calculate $T(\nu)$ from Eq. (5):

$$T(\nu) = \frac{1}{4\pi}\left[\frac{1}{\sin\alpha}(a_1 + \nu\cos\alpha + \nu\sin\alpha\,\mathrm{ctg}\,\beta) + \right.$$
$$+ \frac{1}{\cos\alpha}(b_1 + \nu\sin\alpha + \nu\cos\alpha\,\mathrm{tg}\,\delta) + \frac{\mathrm{ctg}\,\delta}{\cos\alpha}(a_1 +$$
$$+ \nu\cos\alpha + \nu\sin\alpha\,\mathrm{ctg}\,\beta) + \frac{\mathrm{tg}\,\beta}{\sin\alpha}(b_1 +$$
$$\left. + \nu\sin\alpha + \nu\cos\alpha\,\mathrm{tg}\,\delta)\right] \equiv A + \nu B, \tag{7}$$

where

$$A = \frac{1}{4\pi}\left[a_1\left(\frac{1}{\sin\alpha} + \frac{\mathrm{ctg}\,\delta}{\cos\alpha}\right) + \right.$$
$$\left. + b_1\left(\frac{1}{\cos\alpha} + \frac{\mathrm{tg}\,\beta}{\sin\alpha}\right)\right], \tag{8}$$

$$B = \frac{1}{4\pi}(\mathrm{tg}\,\alpha + \mathrm{ctg}\,\alpha + \mathrm{tg}\,\beta + \mathrm{ctg}\,\beta + \mathrm{tg}\,\delta +$$
$$+ \mathrm{ctg}\,\delta + \mathrm{ctg}\,\delta\,\mathrm{tg}\,\alpha\,\mathrm{ctg}\,\beta + \mathrm{tg}\,\beta\,\mathrm{ctg}\,\alpha\,\mathrm{tg}\,\delta). \tag{9}$$

From Eq. (4) we have

$$\int_0^d \frac{d\nu}{T(\nu)} = \int_0^d \frac{d\nu}{A + B\nu} = \frac{1}{B}\ln\left(1 + \frac{B}{A}d\right) \leqslant C^{-1}. \tag{10}$$

Here B and A are determined according to Eqs. (8) and (9) in terms of the data of the problem. Finally we have

$$B^{-1}\ln(1 + BA^{-1}d) \leqslant C^{-1} \leqslant \ln\frac{ab}{a_1 b_1}. \tag{11}$$

Example 1. In order to estimate the error made in calculating according to Eq. (11), we shall consider an example in which $\alpha = \beta = \delta$; $a/a_1 = b/b_1 = \varkappa$. We have

$$A = \frac{1}{4\pi\varkappa}\left[2\frac{a}{b}\sqrt{a^2+b^2} + 2\frac{b}{a}\sqrt{a^2+b^2}\right],$$
$$B = \frac{1}{\pi}(\mathrm{tg}\,\alpha + \mathrm{ctg}\,\alpha) = \frac{1}{\pi}\frac{a^2+b^2}{ab},$$
$$d = \frac{\varkappa - 1}{\varkappa}\frac{\sqrt{a^2+b^2}}{2},$$

$$\frac{1}{B}\ln(1 + BA^{-1}d) = \frac{\pi ab}{a^2+b^2}\ln\varkappa.$$

Therefore,

$$\frac{\pi ab}{a^2+b^2}\ln\varkappa \leqslant C^{-1} \leqslant \ln\varkappa^2 = 2\ln\varkappa. \tag{12}$$

We shall consider the case when the rectangle is close to being a square, i.e., $b/a \approx 1$. Then

$$1.57\ln\varkappa \leqslant C^{-1} \leqslant 2\ln\varkappa. \tag{13}$$

If we take C^{-1} to be the arithmetic mean of the upper and lower bounds, i.e., we put

$$C^{-1} = \frac{2\ln\varkappa + 1.57\ln\varkappa}{2} = 1.78\ln\varkappa, \tag{14}$$

then the relative error will not exceed $(1.78 - 1.57)/$ $/1.57 = 0.14$. Thus, it is convenient to use the formula

$$C^{-1} = \frac{1}{2}\left[\ln\frac{ab}{a_1 b_1} + \frac{1}{B}\ln(1 + BA^{-1}d)\right], \tag{14'}$$

where A and B are determined by Eqs. (8) and (9).

An approximate determination of the capacitance according to Eq. (2) can be made as follows. We take any function $u(x)$, transforming into 0 on S_0 and into 1 on S_1, for which the integral $\int_D |\nabla u(x)|^2\,dV,\ x = (x_1, x_2)$ is calculated. The value obtained is an approximate value of the capacitance, exceeding the true value.

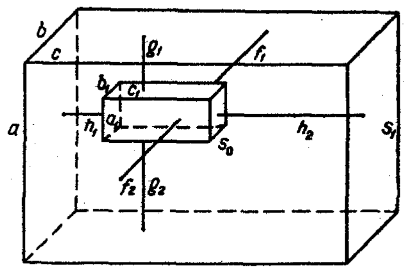

Fig. 3. Parallelepiped within a parallelepiped.

We shall demonstrate another method for evaluating a lower bound of C^{-1}, using the quantities h_1, h_2, g_1, and g_2, $0 \le \nu \le h_1$ (see Fig. 2):

$$T(\nu) = \frac{h_1}{4\pi}\left[\left(\frac{1}{h_1} + \frac{1}{h_2}\right)\left(a\frac{\nu}{h_1} + \right.\right.$$
$$\left. + \left(1 - \frac{\nu}{h_1}\right)a_1\right) + \left(\frac{1}{g_1} + \frac{1}{g_2}\right) \times$$
$$\left.\times\left(b\frac{\nu}{h_1} + \left(1 - \frac{\nu}{h_1}\right)b_1\right)\right] =$$
$$= \frac{h_1}{4\pi}[h(d_1\mu + a_1) + g(d_2\mu + b_1)],$$
$$d_1 \equiv a - a_1;\ d_2 \equiv b - b_1,$$
$$\mu = \nu/h_1;\ h = h_1^{-1} + h_2^{-1};\ g = g_1^{-1} + g_2^{-1}.$$

Hence,

$$C^{-1} \geqslant 4\pi \int_0^{h_1} \frac{d\mu}{h_1(A'\mu + B')} = 4\pi \int_0^1 \frac{d\mu}{A'\mu + B'} = = 4\pi \frac{1}{A'} \ln\left(1 + \frac{A'}{B'}\right), \qquad (10')$$

where

$$A' = hd_1 + gd_2, \quad B' = ha_1 + gb_1.$$

It is appropriate to comment on the matter of reasonable choice (from the viewpoint of minimizing the relative error) of the value of the quantity x, which lies within the limits $Q_0 < x < Q_1$, and takes on any value in the interval (Q_0, Q_1) with equal probability. The desired value $x = 2Q_0Q_1/(Q_1 + Q_0)$ reaches a minimum, since x is then the maximum of the errors $(x - Q_0)/Q_0$ and $(Q_1 - x)/Q_1$. For example, if we use the formula

$$C^{-1} = \frac{2 \cdot 1.57 \ln \varkappa \, 2 \ln \varkappa}{1.57 \ln \varkappa + 2 \ln \varkappa} = 1.76 \ln \varkappa, \qquad (14'')$$

instead of Eq. (14), the relative error will then not exceed

$$\frac{1.76 - 1.57}{1.57} \simeq \frac{2 - 1.76}{2} = 0.12.$$

4. We shall consider the case when the body is situated between parallelepipeds S_0 and S_1. It is convenient to regard the distances d, h_1, h_2, g_1, g_2, f_1, and f_2 as known. We shall estimate an upper bound for the quantity C^{-1} in analogy with Eq. (3). In the three-dimensional case we have

$$C^{-1} \leqslant \frac{1}{r_0} - \frac{1}{r_1},$$

where

$$r_i = \left(\frac{3}{4\pi} V_i\right)^{1/3}, \quad i = 1, 0; \qquad (15)$$

and V_i is the volume closed within the surface S_i.

In our case, $V_1 = abc$, $V_0 = a_1b_1c_1$, and

$$C^{-1} \leqslant \left(\frac{4\pi}{3}\right)^{\frac{1}{3}} \left(\frac{1}{V_0^{1/3}} - \frac{1}{V_1^{1/3}}\right). \qquad (16)$$

We shall calculate $T(\nu)$, having taken $S(\nu)$ to be a parallelepiped situated with respect to the two given parallelepipeds in the same way as the rectangle $S(\nu)$ in Fig. 1 was situated with respect to the rectangles S_0 and S_1. We shall take the parameter ν to be the distance shown in Fig. 3, $0 \leq \nu \leq h_1$. We have

$$T(\nu) = \frac{1}{4\pi}\left\{\left(\frac{h_1}{h_1} + \frac{h_1}{h_2}\right)\left[\frac{a\nu}{h_1} + a_1\left(1 - \frac{\nu}{h_1}\right)\right] \times \right.$$
$$\times \left[\frac{b\nu}{h_1} + b_1\left(1 - \frac{\nu}{h_1}\right)\right] + \left(\frac{h_1}{g_1} + \frac{h_1}{g_2}\right) \times$$
$$\times \left[\frac{b\nu}{h_1} + b_1\left(1 - \frac{\nu}{h_1}\right)\right]\left[\frac{c\nu}{h_1} + c_1\left(1 - \frac{\nu}{h_1}\right)\right] +$$
$$+ \left(\frac{h_1}{f_1} + \frac{h_2}{f_2}\right)\left[\frac{a\nu}{h_1} + a_1\left(1 - \frac{\nu}{h_1}\right)\right] \times$$
$$\left. \times \left[c\frac{\nu}{h_1} + c_1\left(1 - \frac{\nu}{h_1}\right)\right]\right\}.$$

Therefore, putting $\mu = \nu/h_1$, we have

$$C^{-1} \geqslant \int_0^{h_1} d\nu \, T^{-1}(\nu) = 4\pi \int_0^1 \frac{d\mu}{\alpha\mu^2 + \beta\mu + \gamma}, \qquad (17)$$

where

$$\alpha\mu^2 + \beta\mu + \gamma \equiv \left(\frac{1}{h_1} + \frac{1}{h_2}\right)[a\mu + (1-\mu)a_1] \times$$
$$\times [b\mu + (1-\mu)b_1] + \left(\frac{1}{g_1} + \frac{1}{g_2}\right) \times$$
$$\times [b\mu + (1-\mu)b_1][c\mu + (1-\mu)c_1] +$$
$$+ \left(\frac{1}{f_1} + \frac{1}{f_2}\right)[a\mu + a_1(1-\mu)][c\mu + (1-\mu)c_1]. \qquad (18)$$

We shall introduce the notation

$$\frac{1}{h_1} + \frac{1}{h_2} \equiv h, \quad \frac{1}{g_1} + \frac{1}{g_2} \equiv g, \quad \frac{1}{f_1} + \frac{1}{f_2} = f,$$
$$a - a_1 = d_1, \quad b - b_1 = d_2, \quad c - c_1 = d_3. \qquad (19)$$

Then,

$$\alpha = hd_1d_2 + gd_2d_3 + fd_3d_1, \qquad (20)$$
$$\beta = h(d_1b_1 + d_2a_1) + g(d_2c_1 + d_3b_1) + f(d_1c_1 + a_1d_3), \qquad (21)$$
$$\gamma = ha_1b_1 + g\,b_1c_1 + fa_1c_1. \qquad (22)$$

Example 2. Let there be two appropriately located cubes (see Fig. 3) such that

$$h_1 = h_2 = \frac{c - c_1}{2}, \quad g_1 = g_2 = \frac{a - a_1}{2},$$
$$f_1 = f_2 = \frac{b - b_1}{2}, \quad a = b = c,$$
$$\frac{a}{a_1} = \frac{b}{b_1} = \frac{c}{c_1} = \varkappa > 1, \quad h = g = f = \frac{4}{d},$$
$$d = a_1(\varkappa - 1), \quad \alpha = 12(\varkappa - 1)a_1, \quad \beta = 24a_1,$$
$$\gamma = \frac{12a_1^2}{d} = \frac{12}{\varkappa - 1} a_1.$$

Let, for example, $\varkappa = 2$. Then,

$$\left(\frac{4\pi}{3}\right)^{1/3}\left(\frac{1}{a_1} - \frac{1}{2a_1}\right) \geqslant$$
$$\geqslant C^{-1} \geqslant \frac{4\pi}{a_1}\int_0^1 \frac{d\mu}{12\mu^2 + 24\mu + 12} = \frac{\pi}{6a_1},$$
$$\frac{\pi}{6a_1} \leqslant C^{-1} \leqslant \left(\frac{4\pi}{3}\right)^{1/3}\frac{1}{2a_1}.$$
$$\frac{0.52}{a_1} \leqslant C^{-1} \leqslant \frac{0.80}{a_1}. \qquad (23)$$

Therefore, if we put $C^{-1} = (0.80 + 0.52)/2a_1 = 0.66/a_1$, the relative error will not exceed $(0.66 - 0.52)/0.52 = 0.27$. If we choose $0.63/a_1$ for C^{-1}, the relative error does not exceed 0.21. The number 0.63 appears as follows. The error does not exceed the maximum of the numbers $((0.80 - 0.52)/0.52)q$ and $((0.80 - 0.52)/0.80)(1 - q)$, $0 < q < 1$. For $q = 0.39$ both these numbers equal 0.21, and for $q \neq 0.39$ one of them is greater than 0.21.

In conclusion, we compare our value of $C^{-1} = 0.66/a_1$ with the exact known value for a spherical layer enclosed between spheres with radii r_1 and $r > r_1$, for which $C^{-1} = 1/r_1 - 1/r$.

We shall choose r_1 and r such that the volumes of the corresponding spheres are equal, respectively, to the volumes of the cubes with sides a_1 and $a = 2a_1$, i.e., we put $r_1 = a_1(3/4\pi)^{1/3}$, $r = 2r_1$. Then

$$C^{-1} = \frac{1}{r_1} - \frac{r}{r} = \left(\frac{4\pi}{3}\right)^{\frac{1}{3}}\left(\frac{1}{a_1} - \frac{1}{2a_1}\right) = \frac{0.80}{a_1}.$$

As follows from the general considerations presented in Section 1, we obtain the value appearing on the right side of inequality (23).

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